B.Sc. Semester-V Examination, 2022-23

MATHEMATICS (Honours)

Course ID : 52111 **Course Code : SH/MTH/501/C-11**

Course Title : Partial Differential Equations and Applications

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meaning.

1. Answer any **five** questions: 2×5=10

a) When a first order partial differential equation (PDE) is said to be *linear*? Give an example of it.

b) State principle of conservation of linear momentum.

c) Form a partial differential equation by eliminating the arbitrary functions f and g from the relation

$$u = f(x-cy)+g(x+cy), \ c \text{ being constant.}$$

d) Write down the Lagrange's subsidiary equations for the PDE: $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u$ and then find its general solution.

e) Determine all the points at which the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial^2 u}{\partial y^2} - xy\frac{\partial u}{\partial y} = 0 \text{ is elliptic.}$$

f) If a particle moves under a central force, prove that its angular momentum is conserved.

g) Show that at an apse in a central orbit a particle moves perpendicular to the radius vector.

h) Find the family of surfaces orthogonal to the family of surfaces whose partial differential equation is $(y+z)p+(z+x)q = x+y$.

2. Answer any **four** questions: 5×4=20

a) Find the general integral of the partial differential equation (PDE)

$$2y(u-3)\frac{\partial u}{\partial x} + (2x-u)\frac{\partial u}{\partial y} = y(2x-3)$$

and hence find the integral surface passes through the circle $(x-1)^2 + y^2 = 1, \ u = 0$. 3+2

b) Reduce the following partial differential equation to canonical form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + 1 = 0, \ 0 \le x \le 1, \ y > 0$$

and hence obtain its general integral. 3+2

c) Use the method of separation of variables to solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 < x < 4, \ t > 0$$

satisfying the conditions:

$$u(x, 0) = 3\sin \pi x + 6\sin \frac{\pi x}{2}, \quad 0 < x < 4$$
$$u(0, t) = 0, \ u(4, t) = 0, \quad t > 0$$

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d) A particle moves under a force which is always directed towards a fixed point and is equal to $\mu \div (\text{distance})^2$ per unit mass and path is an ellipse. Then show that the velocity v at any time t is $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$, where $2a$ be the length of the major axis. 5

e) Using D' Alembert's formula, obtain the solution of the following initial boundary value problem (IBVP):

$$u_{tt} = 4u_{xx}, \quad 0 < x < \infty, \ t > 0$$
$$u(x, 0) = \cos x, \quad u_t(x, 0) = x, \ 0 < x < \infty$$
$$u_x(0, t) = 0, \quad t > 0$$

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f) A heavy particle slides down the arc of a smooth cycloid whose axis is vertical and vertex downwards. If it starts from rest at a cusp and arrives at the vertex; prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half. 5

3. Answer any **one** question: 10×1=10

a) i) Find the general solution of the following Cauchy problem for the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \ t > 0$$
$$u(x, 0) = f(x), \quad -\infty < x < \infty$$
$$u_t(x, 0) = g(x), \quad -\infty < x < \infty.$$

Hence obtain the solution of the following wave equation

$$u_{tt} = u_{xx}, \ u(x, 0) = \sin x, \ u_t(x, 0) = x, \ -\infty < x < \infty$$

ii) A planet describes an elliptic orbit with the sun at one of its foci under a central force which is always directed towards the sun, find the law of force. (5+2)+3=10

b) i) Solve, by method of separation of variables, the following initial boundary value problem (IBVP):

$$u_t = \alpha^2 u_{xx}, \qquad 0 < x < L,\ t > 0$$
$$u(x, 0) = f(x), \qquad 0 < x < L$$
$$u(0, t) = T_1, \qquad u(L, t) = T_2,\ t > 0$$

where α, T_1 and T_2 are constants.

ii) A particle moves under a central acceleration $\frac{\mu}{r^5}$ (r being the distance from the center of force) and is projected from an apse at a distance a with a velocity $\sqrt{\frac{\mu}{2a^4}}$, show that the equation of the path of the particle is $r = a\cos\theta$. 6+4=10

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